Parameterizations of the Chazy equation

Sarbarish Chakravarty[1] and Mark J Ablowitz[2]

[1] *Department of Mathematics, University of Colorado, Colorado Springs, CO, 80933*

[2] *Department of Applied Mathematics, University of Colorado, Boulder, CO, 80309*

Abstract

The Chazy equation $y''' = 2yy'' - 3y'^2$ is derived from the automorphic properties of Schwarz triangle functions $S(\alpha, \beta, \gamma; z)$. It is shown that solutions y which are analytic in the fundamental domain of these triangle functions, only correspond to certain values of α, β, γ. The solutions are then systematically constructed. These analytic solutions provide all known and one new parametrization of the Eisenstein series P, Q, R introduced by Ramanujan in his modular theories of signature 2, 3, 4 and 6.

1. Introduction

In a series of papers [1]–[3] between 1909 – 1911, J. Chazy considered a class of nonlinear differential equations of the form

$$y''' - 2yy'' + 3y'^{\,2} = \frac{4}{36-k^2}(6y' - y^2)^{\,2}\,, \qquad 0 \leq k \neq 6\,, \tag{1}$$

during the course of his work on the extension of Painlevé's program to equations of third order. The general solutions of these equations, which are now referred to as Chazy class XII, can be parametrized by solutions of appropriate hypergeometric equations for $k > 0$, and the Airy equation when $k = 0$. Furthermore, for $6 < k \in \mathbb{Z}$ the solutions evolve from a generic initial data to form a natural boundary which is closed curve in the complex plane beyond which the solutions can not be analytically continued. For the special cases of $k = 2, 3, 4$ and 5, the associated hypergeometric solutions are algebraic functions classified by Schwarz [4], and leads a 3-parameter family of rational solutions for (1). The focus of this note is on the equation

$$y''' - 2yy'' + 3y'^{\,2} = 0\,, \tag{2}$$

which corresponds to the limiting case $k \to \infty$ of (1), and will be referred to as the Chazy equation throughout this article. Chazy [2] showed that the general solution of (2) also possesses a movable natural boundary by relating the solution $y(z)$ to that of the hypergeometric equation $s(s-1)\chi'' + (\frac{7s}{6} - \frac{1}{2})\chi' - \frac{1}{144}\chi = 0$.

In recent years, it has been shown that the Chazy equation arises in several areas of mathematical physics including magnetic monopoles [5], self-dual Yang-Mills and Einstein equations [6, 7], and topological field theory [8]. In addition, (2) has been derived as special reductions of hydrodynamic type equations [9] as well as stationary, incompressible Prandtl boundary layer equations [10]. These results have renewed interest in the study of the Chazy equation. For example, the $\mathrm{SL}_2(\mathbb{C})$ symmetry of (2) was exploited to systematically derive its general solution in [11]; moreover, in Ref. [12] the group invariance was applied to elucidate the role of its pole singularities, which under suitable perturbations coalesce to the natural boundary. Yet another interest in Chazy equation stems from its well-known connection with the automorphic forms associated with the modular group $\mathrm{SL}_2(\mathbb{Z})$ and its subgroups. The notation $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ for the full modular group (cf. [36]) will be used throughout this article. It is quite significant that the emergence of the Chazy equation (2) in the theory of modular forms and elliptic functions can be traced back to the work of Ramanujan and subsequently in the work of Rankin and others. Here we briefly recall some of these interesting results which were anteceded by Chazy's work, but were apparently not noticed by those researchers.

In 1916, Ramanujan [13], [14, pp 136–162], introduced the functions $P(q)$, $Q(q)$ and $R(q)$ defined for $|q| < 1$ by

$$P(q) := 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\,, \quad Q(q) := 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}\,, \quad R(q) := 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}\,, \tag{3}$$

and proved by using trigonometric series identities that these functions in (3) satisfy the differential relations

$$\delta P = \frac{P^2 - Q}{12}\,, \qquad \delta Q = \frac{PQ - R}{3}\,, \qquad \delta R = \frac{PR - Q^2}{2}\,, \qquad \delta := q\frac{d}{dq}\,. \tag{4}$$

By expressing the system (4) as a single differential equation for $P(q)$, then defining

$$y(z) := \pi i P(q)\,, \qquad q := e^{2\pi i z}\,, \quad \mathrm{Im}(z) > 0\,, \tag{5}$$

one easily recovers the Chazy equation (2) [15]. (Note that in this case the natural boundary corresponds to the unit circle $|q| = 1$ or the real axis $\mathrm{Im}(z) = 0$). Ramanujan's $P(q), Q(q), R(q)$ correspond to the (first three) Eisenstein series associated with the modular group $\Gamma(1)$, but this modern terminology is not needed in the present context. Ramanujan also considered the modular discriminant function

$$\Delta(q) := \frac{Q^3(q) - R^2(q)}{1728} = q\prod_{n=1}^{\infty}(1-q^n)^{24} := \sum_{n=1}^{\infty}\tau(n)q^n\,, \qquad \tau(n) \in \mathbb{Z}\,,$$

he proved as well as conjectured many properties associated with the integer coefficients $\tau(n)$ above, which are referred to as Ramanujan's tau-functions. It follows from the last two equations in (4) that $P(q)$ is simply the logarithmic derivative of $\Delta(q)$, that is

$$P(q) = \frac{\Delta'(q)}{\Delta(q)}\,, \qquad \text{or} \qquad y(z) = \frac{1}{2}\frac{\Delta'(z)}{\Delta(z)}\,. \tag{6}$$

Equation (6) also follows by taking the logarithmic derivative of the infinite product formula for $\Delta(q)$ above, and comparing it with the q-expansion for $P(q)$ in (3). Rankin in [16], showed using properties of modular forms that $\Delta(z)$ satisfies the equation

$$2\Delta''''\Delta^3 - 10\Delta'''\Delta'\Delta^2 - 3\Delta''^{\,2}\Delta^2 + 24\Delta''\Delta'^{\,2}\Delta - 13\Delta'^{\,4} = 0\,,$$

which is homogeneous of degree 4 (in both Δ and its derivatives). Rankin's Δ-equation follows from the Chazy equation (2) with $y(z)$ as in (6). In Ref. [16], Rankin also derives (among others) two more equations which are equivalent to

$$4Q\delta^2 Q - 5(\delta Q)^2 = 960\Delta\,, \qquad 6R\delta^2 R - 7(\delta R)^2 = -3024 Q\Delta\,,$$

which can be also deduced from the Ramanujan equations (4). Moreover, re-expressing the first equation above in terms of $P(q)$ and its derivatives by employing (4) and the definition of $\Delta(q)$, yields (2) once more; while the second equation turns into the differential consequence of the Chazy equation. The first equation for Q above, was also obtained by B. van der Pol, who used it to derive the arithmetical identity [17]

$$\tau(n) = n^2\sigma_3(n) + 60\sum_{m=1}^{n-1}(2n-3m)(n-3m)\sigma_3(m)\sigma_3(n-m)\,, \quad \sigma_k(n) := \sum_{d|n} d^k\,,\ n \in \mathbb{N}\,,$$

relating Ramanujan's tau-functions and the sum-of-divisor function $\sigma_3(n)$. This and such other arithmetical identities follow from equating the q-expansions of both sides of a differential (or polynomial) relation involving the modular functions. We note here that using (5) in the Chazy equation leads to the identity

$$n^2(n-1)\sigma_1(n) + \sum_{m=1}^{n-1} 12m(5m-3n)\sigma_1(m)\sigma_1(n-m) = 0\,,$$

that follows from the q-expansion of $P(q)$ in (3).

Ramanujan extensively studied the properties of his modular functions P, Q, R and established numerous identities involving them [14, 18]. Quite remarkably, he like Chazy, also employed the theory of hypergeometric functions to establish an implicit parametrization of the functions P, Q, R, which plays a crucial role in the proofs of Ramanujan's modular identities. In his second notebook [18], Ramanujan considered the hypergeometric function ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)$ related to the complete elliptic integral of the first kind by (see e.g., [35, 29])

$$K(x) := \int_0^{\pi/2} \frac{dt}{\sqrt{1 - x\sin^2 t}} = \frac{\pi}{2}\, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; x)\,,$$

and gave the following implicit parametrization of the functions P, Q, R

$$P(q) = (1-5x)\chi^2 + 12x(1-x)\chi\chi'\,, \quad Q(q) = (1+14x+x^2)\chi^4\,, \quad R(q) = (1+x)(1-34x+x^2)\chi^6\,, \tag{7}$$

where $\chi(x) := {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)$ and the nome q is defined by

$$q = e^{-u}\,, \qquad u := \pi\, \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}\,.$$

Equation (7) originates from Jacobi's work on elliptic functions [19]. The derivation of (7) involves expressing P, Q, R as logarithmic derivatives of the quotients of theta functions and application of the remarkable Jacobi-Ramanujan inversion formula

$$\vartheta_3(0|q) := 1 + 2\sum_{n=1}^{\infty} q^{n^2} = \sqrt{\chi(x)}\,.$$

It is worth pointing out here that in Refs. [1, 3], Chazy had also noted the equivalence between (2) and a system of three first order equations introduced by G. Darboux [20] in 1878 (see Section 2). This first order system is different from the Ramanujan system (4) even though its solutions were given by G. Halphen [21] in terms of logarithmic derivatives of the null theta functions $\vartheta_2, \vartheta_3, \vartheta_4$. Each of the null theta functions satisfies Jacobi's nonlinear equation

$$(\theta^2\theta''' - 15\theta\theta'\theta'' + 30\theta'^3)^2 = 32(\theta\theta'' - 3\theta'^2)^3 + \pi^2\theta^{10}(\theta\theta'' - 3\theta'^2)^2 = 0\,,$$

for $\theta(z) := \vartheta_i(0|q)\,,\, i = 2...4$ and where $q = e^{\pi i z}$ [22]. Ramanujan relied heavily on the parametrization (7) and the inversion formula to develop his theory of modular equations involving functional relations among complete elliptic integrals at different arguments. Interestingly, he also proposed alternative parametrizations for $P(q), Q(q), R(q)$ in terms of other hypergeometric functions (see Section 4), and where the appropriate nomes q can be expressed by

$$q_r = e^{-u_r}\,, \qquad u_r := \frac{\pi}{\sin(\frac{\pi}{r})}\, \frac{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; 1-x)}{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; x)}\,, \qquad r = 2, 3, 4, 6\,. \tag{8}$$

The index r is referred to as the *signature* of Ramanujan's theories. The case $r = 2$ corresponds to Ramanujan's original theory of modular equations while signatures 3,4 and 6 correspond to Ramanujan's *alternative* theories. He stated the results in his alternative

theories without proof [18]. Proofs were constructed much later [23, 24] and is now an important topic of research [25]. Indeed a unified framework of Ramanujan's modular equations in different signatures based on the more contemporary theory of modular forms, elliptic surfaces and Gauss-Manin connections was proffered only recently in Ref [26].

In this paper, we derive the parametrization for the functions P, Q, R in Ramanujan's original and alternative theories, as well as some new ones (see cases (1), (2) in Table 1) via the parametrization of the Chazy solution $y(z)$ in terms of Schwarz triangle and hypergeometric functions. Our approach is not number theoretic, but rather based on the theory of Fuchsian differential equations and the action of its projective monodromy group on certain differential polynomials. Section 2 provides some necessary background on Fuchsian equations and their role in the conformal mappings of the complex plane to triangular domains T bounded by circular arcs. The triangle functions introduced by Schwarz form the natural coordinates on T, and which remain invariant under the group of automorphisms induced by the projective monodromy groups of the Fuchsian equations. The relationship between the triangle functions and the Chazy equation is established in Section 3. In section 4, we give the explicit parametrization of the Chazy solution $y(z)$ in terms of the appropriate hypergeometric functions, thereby connecting our results to those of Ramanujan's parametrization in his theories of modular equations. Along these lines, we also note the work in Ref. [27] which utilizes arguments motivated by the Lie symmetry analysis of (4) and related differential equations to obtain various parametrizations for the functions P, Q, R. The approach in this note is different from that in Ref. [27].

2. Conformal mapping and triangle functions

The mapping properties of Fuchsian differential equations play a significant role in the theory of automorphic functions. In particular, H. A. Schwarz in 1873 carried out an exhaustive study of the conformal maps induced by ratio of solutions of hypergeometric equations and the so called triangle functions [4]. This beautiful theory has since been developed significantly and is treated in numerous monographs. The brief overview presented in this section closely follow the texts [28] and [29].

A second order, Fuchsian differential equation with three regular singular points in the complex plane can be cast into the form

$$u'' + \frac{V(s)}{4}\, u = 0\,, \qquad\qquad V(s) = \frac{1-\alpha^2}{s^2} + \frac{1-\beta^2}{(s-1)^2} + \frac{\alpha^2+\beta^2-\gamma^2-1}{s(s-1)}\,, \tag{9}$$

where α, β, γ are the exponent differences (for any pair of linearly independent solutions) prescribed at the singular points 0, 1 and ∞, respectively. (Note that f' indicates derivation with respect to the argument of the function f throughout this article). The ratio $z(s)$ of any two linearly independent solutions u_1, u_2 of (9) is a multi-valued function branched at the regular singular points, and satisfies the Schwarzian differential equation

$$\{z, s\} = \frac{V(s)}{2}\,, \qquad\qquad \{z, s\} := \frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2\,. \tag{10}$$

The monodromy group $G \subset \mathrm{GL}_2(\mathbb{C})$ resulting from the analytic extensions of the pair (u_1, u_2) along all possible closed loops through an ordinary point s_0, is determined (modulo conjugation) by the exponent differences. G acts projectively on the ratio $z(s)$ via

fractional linear transformations

$$z \to \gamma(z) := \frac{az+b}{cz+d}\,, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G\,.$$

Since both γ and $\lambda\gamma$ yield the same fractional linear transformation for any complex $\lambda \neq 0$, the projectivized monodromy group is the quotient group $\Gamma \cong G/\lambda I \subseteq \mathrm{PSL}_2(\mathbb{C})$ where I is the 2×2 identity matrix. Note that both z and $\gamma(z)$ satisfy the same equation (10) due to the invariance property of the Schwarzian derivatives: $\{z, s\} = \{\gamma(z), s\}$. A special class of solutions of (10) was extensively investigated by Schwarz who considered the parameters in $V(s)$ to be real and $0 < \alpha, \beta, \gamma < 1$. If $V(s)$ is to be further restricted such that $\alpha+\beta+\gamma < 1$, then a branch of $z(s)$ maps the upper-half s-plane (Im $s \geq 0$) onto a hyperbolic triangle T in the extended complex plane, bounded by three circular arcs which enclose interior angles $\alpha\pi$, $\beta\pi$, and $\gamma\pi$ at the vertices $z(0)$, $z(1)$, and $z(\infty)$. By the Schwarz reflection principle, the analytic extension of this branch to the lower-half plane across a line segment between any two branch points, maps the lower-half plane to an adjacent triangle T$'$ that is the image of T under reflection across the circular arc which forms their common boundary. Continuing this process, the complete set of branches of $z(s)$ maps the s-plane onto a Riemann surface spread over the z-plane consisting of an infinite number of circular triangles obtained by inversions across the boundaries of T and its images. The necessary and sufficient condition that this Riemann surface is a plane region D (being the uniform covering of non-overlapping triangles) is that the exponent differences α, β, γ be either zero or reciprocals of positive integers. In this case, the inverse $s(z)$ is a single-valued, meromorphic, automorphic function whose automorphism group is the projective monodromy group Γ defined above. That is, $s(\gamma(z)) = s(z)\,, \gamma \in \Gamma$. In this setting of conformal mapping, Γ is a discrete subgroup of $PSL(2, \mathbb{R})$, and turns out to be the group of fractional linear transformations generated by an even number of reflections across the boundaries of the circular triangles. More precisely, let $r_\alpha, r_\beta, r_\gamma$ be the reflections across the sides opposite to the vertices $z(0)$, $z(1)$, and $z(\infty)$ of T. Then the automorphism group is generated by the elements $R_\alpha = r_\beta r_\gamma$, $R_\beta = r_\alpha r_\gamma$ and $R_\gamma = r_\alpha r_\beta$ which are rotations about each vertex by $2\pi\alpha, 2\pi\beta$ and $2\pi\gamma$, and satisfy

$$R_\alpha^{1/\alpha} = R_\beta^{1/\beta} = R_\gamma^{1/\gamma} = R_\alpha R_\beta R_\gamma = 1\,.$$

A vertex with a nonzero (interior) angle π/m, $m \in \mathbb{Z}^+$ is called an elliptic fixed point of order m, whereas a vertex with zero angle is called a parabolic fixed point of the group.

In its domain of existence D, the only possible singularities of $s(z)$ and its derivatives are located only at the vertices where $s(z)$ takes the value of 0, 1, or ∞. Due to the automorphic property, it is sufficient to consider the function $s(z)$ on the triangle T. Let $z = z_0$ be a vertex of the triangle T such that $s(z_0) = s_0 \in \{0, 1, \infty\}$ is a regular singular point of the Fuchsian equation (9) with exponent difference $\mu \in \{\alpha, \beta, \gamma\}$. The behavior of the $s(z)$ near a vertex z_0 depends on whether z_0 is an elliptic or a parabolic fixed point of Γ.

(i) If $\mu = 1/m\,, m \in \mathbb{Z}^+$, then z_0 is an elliptic fixed point of order m. In this case, a pair of fundamental solutions of (9) in the neighborhood of $s = s_0 \neq \infty$ are of the form

$$u_1(s) = (s - s_0)^{(1+\mu)/2}\psi_1(s)\,, \qquad u_2(s) = (s - s_0)^{(1-\mu)/2}\psi_2(s)\,,$$

where $\psi_i(s)$, $i = 1, 2$ admit convergent power series in the neighborhood of $s = s_0$, and $\psi_i(s_0) \neq 0$. By taking appropriate linear combinations of the solutions u_1 and u_2, $z(s)$ is

defined via

$$z - z_0 = \frac{u_1}{u_2} = (s - s_0)^\mu \psi(s) \,,$$

where $\psi(s)$ is analytic near $s = s_0$ and $\psi(s) \neq 0$. The inverse function is single-valued, and given by

$$s(z) = s_0 + (z - z_0)^m \phi_1(z) \,, \qquad \phi_1(z_0) \neq 0 \,, \tag{11a}$$

where $\phi(z)$ is analytic near $z = z_0$. Thus, $s - s_0$ has a zero of order m at $z = z_0$. If $s_0 = \infty$, then by making the transformation $s' = 1/s$ in (9) one finds in a similar manner as above that $s(z)$ has a pole of order m at $z = z_0$. That is,

$$s(z) = (z - z_0)^{-m} \phi_2(z) \,, \qquad \phi_2(z_0) \neq 0 \,, \tag{11b}$$

and $\phi_2(z)$ is analytic in the neighborhood of $z = z_0$. (ii) When z_0 is a parabolic vertex, $\mu = 0$. In this case, a pair of linearly independent solutions of (9) in a neighborhood of $s = s_0 \neq \infty$ is given by

$$u_1(s) = (s - s_0)^{1/2} \psi_3(s) \,, \qquad u_2(s) = [k \log(s - s_0) + \psi_4(s)] u_1(s) \,,$$

where $\psi_3(s_0) \neq 0$, $\psi_4(s_0) = 0$, k is a constant, and both $\psi_3(s), \psi_4(s)$ admit convergent power series in the neighborhood of $s = s_0$. In terms of u_1, u_2, the function $z(s)$ can be defined as

$$2\pi i z = \frac{u_2}{u_1} \qquad \text{or} \qquad \frac{2\pi i}{z - z_0} = \frac{u_2}{u_1} \,,$$

depending on whether $z_0 = \infty$, or a finite vertex in the extended z-plane. From above, the inverse function $s(z)$ can be expressed as a power series

$$s(z) = s_0 + \sum_{n=1}^{\infty} c_n q^n \,, \qquad q := e^{2\pi i z/k} \left(\text{or } q := e^{\frac{2\pi i}{k(z - z_0)}} \right) \,, \qquad c_n \in \mathbb{C} \,, \tag{12a}$$

in the uniformizing variable q. Thus, $s(z)$ is a single-valued function of z, holomorphic at $q = 0$. If $s_0 = \infty$, then a similar analysis as above can be carried out by making the transformation $s' = 1/s$, and by introducing the same local uniformizer q as in (12a). In this case, $s(z)$ has a pole at $q = 0$, and the q-expansion

$$s(z) = \frac{d}{q} + \sum_{n=0}^{\infty} d_n q^n \,, \qquad d,\, d_n \in \mathbb{C} \,. \tag{12b}$$

A pair of adjacent triangles T and T′ form the fundamental region X of the automorphism group Γ whose action on X tessellates the region D. The inverse function $s(z)$, which maps X to the entire extended s-plane, generates the function field (over $\mathbb{C}$) of X. The boundary of D in the z-plane, is a Γ-invariant circle which is orthogonal to (all three sides of) the triangle T and all its reflected images. This orthogonal circle is the set of limit points for the automorphic group Γ, it is a dense set of essential singularities forming a *natural boundary* for the function $s(z)$. In its domain of existence D, the only possible singularities of $s(z)$ are poles which correspond to the vertices where $s(z) = \infty$. Thus, Γ is a Fuchsian group of the first kind (see e.g., [28], Sec. 30 for a definition) and $s(z)$ is a simple automorphic function of Γ. Fuchsian groups associated with differential equations (9) with three regular singular points are referred to as triangle groups and

$s(z) := S(\alpha, \beta, \gamma; z)$ are called Schwarz triangle functions. It follows from (10) that the inverse function $s(z)$ satisfies the following third order nonlinear equation

$$\{s, z\} + \frac{s'^2}{2} V(s) = 0 \,. \tag{13}$$

Conversely, when the parameters α, β, γ in $V(s)$ are either zero or reciprocals of positive integers, a three-parameter family of solution of (13) is obtained as the inverse of the ratio

$$z(s) = \frac{Au_1(s) + Bu_2(s)}{Cu_1(s) + Du_2(s)}, \qquad A, B, C, D \in \mathbb{C}, \quad AD - BC = 1, \tag{14}$$

where u_1 and u_2 are linearly independent solutions of (9). The solution is single-valued and meromorphic inside a disk in the extended z-plane, and can not be continued analytically across the boundary of the disk. This boundary is movable as its center and radius are completely determined by the initial conditions which depend on the complex parameters A, B, C, D.

A number of nonlinear differential equations whose solutions possess movable natural boundaries, can be solved by first transforming them into a Schwarzian equation (13) and then following the *linearization* scheme described above. We briefly recount some examples of such nonlinear differential equations and their relations to Schwarz triangle functions. In 1881, Halphen considered a slightly different version [30, pp 1405, Eq (5)] of the following nonlinear differential system

$$\begin{aligned} w_1' &= -w_2 w_3 + w_1(w_2 + w_3) + \tau^2 \,, \\ w_2' &= -w_3 w_1 + w_2(w_3 + w_1) + \tau^2 \,, \\ w_3' &= -w_1 w_2 + w_3(w_1 + w_2) + \tau^2 \,, \end{aligned} \tag{15}$$

$$\tau^2 = \alpha^2(w_1 - w_2)(w_2 - w_3) + \beta^2(w_2 - w_1)(w_1 - w_3) + \gamma^2(w_3 - w_1)(w_2 - w_3) \,,$$

for functions $w_i(z) \neq w_j(z)$, $i \neq j$, $i, j \in \{1, 2, 3\}$, and constants α, β, γ. Halphen presented the solutions of (15) in terms of hypergeometric functions. More recently, the authors found that (15) arises as a symmetry reduction of self-dual Yang-Mills equations, and called it the generalized Darboux-Halphen (gDH) system [31, 32]. If in fact, $w_1(z)$, $w_2(z)$, $w_3(z)$ are parametrized in terms of a single function $s(z)$ (and its derivatives) as

$$w_1 = \frac{1}{2}\left[\log\left(\frac{s'}{s}\right)\right]', \quad w_2 = \frac{1}{2}\left[\log\left(\frac{s'}{s-1}\right)\right]', \quad w_3 = \frac{1}{2}\left[\log\left(\frac{s'}{s(s-1)}\right)\right]', \tag{16}$$

then $s(z)$ is a solution of (13), where the constants α, β, γ in $V(s)$ of (13) are the same as those appearing in τ^2 of (15). The special case $\alpha = \beta = \gamma = 0$ of the gDH system corresponds to the "classical" Darboux-Halphen (DH) system which is equation (15) with $\tau^2 = 0$. This equation originally appeared in Darboux's work of triply orthogonal surfaces on $\mathbb{R}^3$ in 1878 [20], and its solution was subsequently given by Halphen [21] in 1881. The variables w_i associated with the DH system can be parametrized as in (16) by the triangle function $s(z) = S(0, 0, 0; z)$; the latter is related to the elliptic modular function $\lambda(z) := \vartheta_2^4(0|z)/\vartheta_3^4(0|z)$, expressed in terms of null theta functions. Chazy showed that the function $y(z) := 2(w_1 + w_2 + w_3)$ satisfies (2) introduced in Section 1. Chazy [3] also noted that besides the elliptic modular function $\lambda(z)$, the solution to (2) can also

be given in terms of the triangle function $S(\frac{1}{2}, \frac{1}{3}, 0; z)$ which is the same as the modular J-function for the group $\mathrm{SL}_2(\mathbb{Z})$. In more recent work Bureau [33] in 1987, investigated a class of third order nonlinear equations, and expressed their general solutions in terms of the Schwarz triangle functions $s := S(\alpha, \beta, \gamma; z)$. In particular, Bureau's class includes the Chazy equation (2).

It is natural to inquire whether the solution of the Chazy equation (2) admits parametrization in terms of *other* triangle functions besides $S(0, 0, 0; z)$ and $S(\frac{1}{2}, \frac{1}{3}, 0; z)$. One motivation of the present work is to address this question and to investigate the possible linearizations of the Chazy equation (2) via solutions of the Fuchsian equation (9) with parameters $\{\alpha, \beta, \gamma\}$ other than $\{\frac{1}{2}, \frac{1}{3}, 0\}$ or $\{0, 0, 0\}$. In the following we outline a method to systematically derive the Chazy equation from the solution of the Schwarz equation (13) for appropriate values of the parameters α, β, γ. Our construction utilizes the transformation properties of $s(z)$ and its derivatives under the automorphism group Γ and selects those triangle functions which provide a natural parametrization of the Chazy solution $y(z)$ that is holomorphic in their domain of existence D.

Triangle functions and the Chazy equation

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian triangle group with α, β, γ either zero or a reciprocal of positive integers, as in the previous section, and let $s(z)$ be a simple automorphic function of Γ defined on a domain D of the complex plane. A meromorphic function f on D is called a *automorphic form* of *weight k* for Γ if

$$f(\gamma(z)) = (cz + d)^k f(z)\,, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\,, \qquad \gamma(z) = \frac{az + b}{cz + d}$$

for all $z \in \mathrm{D}$. If $k = 0$, then f is called a *automorphic function* on Γ and is a rational function of $s(z)$. Consider a gDH system (15) for Γ where the gDH variables w_i, $i = 1, 2, 3$ are parametrized as in (16). From the automorphic property: $s(\gamma(z)) = s(z)$, it follows that $s'(\gamma(z)) = (cz + d)^2 s'(z)$, and that

$$w_i(\gamma(z)) = (cz + d)^2 w_i(z) + c(cz + d)\,, \qquad \gamma \in \Gamma\,.$$

That is, $s'(z)$ is a weight 2 automorphic form, whereas the w_i are called *quasi*-automorphic forms of weight 2. Define in terms of the gDH variables, the following function

$$y(z) = a_1 w_1 + a_2 w_2 + a_3 w_3\,, \tag{17}$$

on D, where the coefficients a_i are constants. The objective of this section is to determine an autonomous differential equation which is a polynomial in y and its derivatives.

It follows from the transformation property of the w_i above, that $y(z)$ transforms under the action of Γ as

$$y(\gamma(z)) = (cz + d)^2 y(z) + pc(cz + d)\,, \qquad \gamma \in \Gamma\,, \tag{18}$$

where $p = a_1 + a_2 + a_3$ is called the coefficient of *affinity* of the quasi-automorphic form $y(z)$. Furthermore, a sequence of automorphic forms can be constructed on the ring of differential polynomials of $y(z)$ as follows:

Lemma 1. *Let $y(z)$ be a quasi-automorphic form of Γ with affinity coefficient p, then $f_2 = y' - y^2/p$ is an automorphic form of weight 4, and $f_{n+1} = f_n' - (2n/p)yf_n$, $n \geq 2$ are automorphic forms of weight $2n+2$ of Γ.*

Proof. Differentiating (18), one finds that

$$y'(\gamma(z)) = (cz+d)^2\left((cz+d)^2y(z) + pc(cz+d)\right)' = (cz+d)^4\left(y'(z) + \frac{2cy(z)}{cz+d} + \frac{pc^2}{(cz+d)^2}\right)$$

and also from (18),

$$y(\gamma(z))^2 = (cz+d)^4\left(y(z)^2 + \frac{2pcy(z)}{cz+d} + \frac{p^2c^2}{(cz+d)^2}\right).$$

Then, by combining the two expressions above yields the desired transformation property for f_2. The rest follows by induction and use of (18). $\square$

The parametrization (16) of the gDH variables and (17) yield the following expression for $y(z)$ in terms of $s(z)$ and its derivatives

$$y(z) = \frac{p}{2}\frac{\phi'(z)}{\phi(z)}, \qquad \phi(z) = \frac{s'(z)}{(s-1)^{b_1}s^{b_2}}, \tag{19}$$

with $b_1 = 1 - a_1/p$ and $b_2 = 1 - a_2/p$. Then all the higher derivatives of y can be expressed also in terms of $s(z), s'(z)$ and $s''(z)$ by differentiating (19) successively and using the Schwarz equation (13) for $s(z)$. In particular, the differential polynomials f_k introduced in Lemma 1 are given by

$$f_n = -\frac{p}{2}V_n(s)s'(z)^n, \qquad V_{n+1}(s) = V_n'(s) + nq(s)V_n(s), \quad n \geq 2, \tag{20}$$

where $V_n(s)$ are rational functions of s defined recursively from

$$V_2(s) = \tfrac{1}{2}V(s) + q'(s) + \tfrac{1}{2}q(s)^2, \qquad q(s) = \frac{b_1}{s-1} + \frac{b_2}{s}, \tag{21}$$

and where $V(s)$ is given in (9). Combining Lemma 1 with (20) for $n = 2, 3, 4$, and with $f_2 \neq 0$, one can construct the following rational expressions in $y(z), y'(z), y''(z)$ and $y'''(z)$

$$\frac{f_3^2}{f_2^3} = -\frac{2}{p}\frac{V_3(s)^2}{V_2(s)^3}, \qquad \frac{f_4}{f_2^2} = -\frac{2}{p}\frac{V_4(s)}{V_2(s)^2}, \tag{22}$$

which are rational in $s(z)$. Eliminating s from the two equations in (22) leads, in the general case, to a third order, nonlinear equation, rational in $y(z)$ and its derivatives, and which depends explicitly on the parameters p, b_1, b_2 (equivalently, a_1, a_2, a_3), and α, β, γ. For suitable choices of these parameters, the third order equation constructed this way can be identified with a large set of nonlinear differential equations, and whose solutions $y(z)$ are given via (19), in terms of the solutions of the Schwarzian equation (13). Furthermore, it follows from Section 2 that all such solutions will admit a natural boundary, and will be meromorphic in the domain of existence D. Of these third order equations, only the special case of the Chazy equation will be considered in this note, leaving the general classification problem for a future work.

The Chazy equation (2) can be expressed simply as a polynomial equation in terms of the automorphic forms f_2 and f_4. Indeed, from Lemma 1

$$f_4 = y''' - \frac{12}{p} y y'' + \frac{18}{p} y'^2 - \frac{24}{p} f_2^2 \,.$$

Then imposing the constraint

$$p := a_1 + a_2 + a_3 = 6 \,,$$

on the coefficients a_i in (17) in the above expression for f_4, yields the alternative expression: $f_4 + 4f_2^2 = 0$ for the Chazy equation (2). Note that this expression is equivalent to the vanishing of a certain automorphic form of weight 8 associated with the Fuchsian group Γ (see Section 4). Then the second equation in (22) implies that the Chazy equation must be equivalent to the condition

$$V_4 = 12 V_2^2 \,, \tag{23}$$

which needs to hold for *all* s. Condition (23) imposes certain restrictions on the parameters b_1, b_2, and α, β, γ appearing in $V_2(s), V_4(s)$. However, we impose further constraints on the parameters b_1, b_2 by demanding that the meromorphic function $y(z)$ be in fact, *holomorphic* in its domain of existence D. The reason for this additional condition is motivated from the known result (see e.g. [2]) that the general solution $y(z)$ of the Chazy equation obtained via the triangle function $S(\frac{1}{2}, \frac{1}{3}, 0; z)$ is analytic on D although there exists particular (2-parameter family) solutions that are meromorphic but do not possess a natural boundary. Since one of the main objectives of this paper is to obtain parametrizations of the Chazy equation in terms of triangle functions, we impose the holomorphicity of $y(z)$ a priori; then this leads to the specific choices for the parameters $\{\alpha, \beta, \gamma\}$ for the triangle functions which parametrize the Chazy solution $y(z)$. These algebraic conditions will be systematically investigated next.

In order to determine whether $y(z)$ defined in (17) is holomorphic on D, it is necessary to analyze the singularities of the gDH variables w_i given in terms of $s(z)$ and its derivatives in (16). It follows from the conformal mapping theory discussed in Section 2 that it is sufficient to examine the behavior of the function $s(z)$ and its derivatives near the vertices $z(0), z(1)$, and $z(\infty)$ of the fundamental triangle T. The Schwarz reflection principle and the automorphic property then ensure that $s(z)$ will have the same behavior at the vertices of the reflected triangles in D.

Lemma 2. *Let $s(z) = S(\alpha, \beta, \gamma; z)$ be the Schwarz triangle function of a Fuchsian group Γ with fundamental triangle T whose interior angles at the vertices $z(0), z(1), z(\infty)$ are respectively, $\alpha\pi, \beta\pi$ and $\gamma\pi$. Furthermore, let the gDH variables w_1, w_2, w_3 be given by $s(z)$ and its derivatives as in* (16)*. If $\{\alpha, \beta, \gamma\} = \{1/p_1, 1/p_2, 1/p_3\}$, $p_j \in \mathbb{Z}^+$, $j = 1, 2, 3$, then w_1, w_2, w_3 have first order poles at each of the vertices with the following residue scheme:*

$$\operatorname{Res}_{z=z(0)}\{w_1, w_2, w_3\} = \{-\tfrac{1}{2}, \tfrac{1}{2}(p_1 - 1), -\tfrac{1}{2}\} \,, \qquad \operatorname{Res}_{z=z(1)}\{w_1, w_2, w_3\} = \{\tfrac{1}{2}(p_2 - 1), -\tfrac{1}{2}, -\tfrac{1}{2}\} \,,$$
$$\operatorname{Res}_{z=z(\infty)}\{w_1, w_2, w_3\} = \{-\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}(p_3 - 1)\}.$$

Proof. Direct computation of w_1, w_2, w_3 in (16) using equations (11a) and (11b). $\square$

It follows from Lemma 2 that only possible singularities of $y(z)$ defined by (17) in the domain D, are first order poles at the vertices corresponding to the elliptic fixed points of Γ. (Note however that $y(z)$ has a dense set essential singularities at the boundary of D as is the case for the function $s(z)$). Therefore, $y(z)$ will be holomorphic in the entire domain D if the coefficients a_i in (17) can be chosen such that the residue at each pole vanishes. Consider first the case $\{\alpha, \beta, \gamma\} = \{1/p_1, 1/p_2, 1/p_3\}$, where $p_j \in \mathbb{Z}^+$. From part (i) of Lemma 2, the condition that the residue of $y(z)$ vanishes at each pole $z(0), z(1), z(\infty)$, leads to a set of homogeneous, linear equations for a_i, namely,

$$\begin{aligned} a_1 + (1-p_1)a_2 + a_3 &= 0\,, \\ (1-p_2)a_1 + a_2 + a_3 &= 0\,, \\ a_1 + a_2 + (1-p_3)a_3 &= 0\,. \end{aligned} \tag{24}$$

For nontrivial solutions a_i, the vanishing of the determinant of the coefficient matrix in (24) implies that $1/p_1 + 1/p_2 + 1/p_3 = \alpha + \beta + \gamma = 1$, while the conformal mapping of the upper-half s-plane onto the hyperbolic triangle requires that $\alpha + \beta + \gamma < 1$. Therefore, only two out of three equations in (24) can be used to set the residue equal to zero. Thus, for $y(z)$ to be holomorphic on D, one of the three vertices must be a parabolic vertex at the boundary of D instead of an elliptic vertex with a pole singularity. Hence, we have the following result.

Proposition 3. *If a solution $y(z)$ of the Chazy equation* (2) *is parametrized by the triangle function $s(z) = S(\alpha, \beta, \gamma; z)$ defined on a domain D, then a necessary condition for $y(z)$ to be holomorphic on D is that at least one of the parameters α, β, γ be zero.*

Here we emphasize that it is indeed possible for the solutions $y(z)$ of other nonlinear equations obtained from the general construction outlined below (22), to be meromorphic in its domain of existence D. For example, the general solutions of (1) for $6 < k \in \mathbb{Z}$ are given in terms of the triangle functions $S(\frac{1}{2}, \frac{1}{3}, \frac{1}{k}; z)$, and are meromorphic in D with poles at the elliptic vertices which have interior angles π/k [2]. We will consider the parametrization of (1) and other equations in a future study.

According to Proposition 3, at least one of the vertices of the fundamental triangle T must be a parabolic fixed point of the automorphism group Γ. Thus there are three distinct cases which are considered below. For reasons that will be clear in Section 4, $z(0)$ is always chosen to be a parabolic vertex in each of these cases so that the corresponding triangle functions are of the form $S(0, \beta, \gamma; z)$.

2 elliptic and 1 parabolic vertices: As indicated above, it suffices to choose $z(0)$ to be the parabolic vertex, and $z(1), z(\infty)$ as the elliptic vertices so that $\{\alpha, \beta, \gamma\} = \{0, 1/p_2, 1/p_3\}$. All other sub cases can be generated from permutations of $\{z(0), z(1), z(\infty)\}$. Requiring the residue of $y(z)$ to be zero at each of the poles $z(1), z(\infty)$ yields the last two equations of (24). From these, one easily deduces that $a_1 = p/p_2 = \beta p$ and $a_3 = p/p_3 = \gamma p$ so that $a_2/p = 1 - \beta - \gamma$. Moreover, the parameters b_1, b_2 in (19) are given by $b_1 = 1 - a_1/p = 1 - \beta$ and $b_2 = 1 - a_2/p = \beta + \gamma$. Equations (21) and (20) then deliver the explicit forms for $V_2(s)$ and $V_4(s)$, which depend on the remaining parameters

β, γ. They are given by

$$V_2 = \frac{A}{s^2(s-1)}\,, \qquad \text{with} \qquad A = \frac{-\,(1-\beta-\gamma)^2}{2} \qquad\qquad \text{and}\,,$$

$$V_4 = A\,\frac{(1-2\gamma)(1-3\gamma)s^2 + [(1-\beta-\gamma)(12\gamma-7)+(1-2\gamma)]s + 6(1-\beta-\gamma)^2}{s^4(s-1)^3}\,.$$

Finally, inserting the above expressions in (23) and requiring that the resulting expression

$$s[(1-2\gamma)(1-3\gamma)s + (1-\beta-\gamma)(6\gamma-6\beta-1) + (1-2\gamma)] = 0$$

holds for all $s \notin \{0, 1, \infty\}$, give rise to two distinct solutions:

$$\{\alpha, \beta, \gamma\} = \{0, \tfrac{1}{2}, \tfrac{1}{3}\}\,, \qquad \{\alpha, \beta, \gamma\} = \{0, \tfrac{1}{3}, \tfrac{1}{3}\}\,.$$

1 elliptic and 2 parabolic vertices: Again, without any loss of generality, the elliptic vertex can be chosen as $z(1)$ such that $\{\alpha, \beta, \gamma\} = \{0, 1/p_2, 0\}$. From Lemma 2, $y(z)$ is holomorphic in T except at the vertex $z(1)$ where it has a simple pole. Vanishing of the residue at the pole $z(1)$ gives the second equation in (24), which implies that $a_1 = p/p_2 = \beta p$ and $b_1 = 1 - a_1/p = 1 - \beta$. In this case, b_2 and β are the two remaining parameters in $V_2(s)$ and $V_4(s)$, which take the forms

$$V_2 = \frac{(1-b_2)^2}{2s^2} + \frac{B}{s(s-1)}\,, \qquad V_4 = \frac{N(s)}{s^4(s-1)^3}\,,$$

where $B := (1-\beta)(2b_2-\beta-1)/2$ and $N(s)$ is a cubic polynomial whose coefficients depend on β, b_2. Then (23) implies that $C_0 - C_1 s(s-1) = 0$ for all $s \notin \{0, 1, \infty\}$, where

$$C_0 = \frac{(1-\beta)(1-2\beta)(1-3\beta)}{2}\,, \quad C_1 = (1-\beta)(2-3\beta)(1-b_2)^2 + B[6(1+\beta^2) - 5(b_2+\beta)]\,,$$

and B is defined above. Setting $C_0 = 0 = C_1$, three distinct solutions are found:

$$(i)\,\{\alpha, \beta, \gamma\} = \{0, \tfrac{1}{2}, 0\}\,, \quad b_2 \in \{\tfrac{2}{3}, \tfrac{5}{6}\}; \qquad (ii)\,\{\alpha, \beta, \gamma\} = \{0, \tfrac{1}{3}, 0\}\,, \quad b_2 \in \{\tfrac{1}{2}, \tfrac{5}{6}\};$$
$$(iii)\,\{\alpha, \beta, \gamma\} = \{0, \tfrac{2}{3}, 0\}\,, \quad b_2 = \tfrac{5}{6}\,.$$

Notice that in case (iii), β is *not* reciprocal of a positive integer, hence $S(0, \frac{2}{3}, 0; z)$ is not a single valued function of z.

3 parabolic vertices: Here, $\{\alpha, \beta, \gamma\} = \{0, 0, 0\}$ so that $y(z)$ is holomorphic on D for any choice of the parameters a_i. The functions $V_2(s)$ and $V_4(s)$ depend on the parameters b_1, b_2 which are determined from (23) by requiring that $y(z)$ satisfies the Chazy equation (2). The calculations are tedious but similar to the previous two cases, and there are two distinct solutions: $b_1 = b_2 = \frac{2}{3}$, and $b_1 = b_2 = \frac{5}{6}$.

The triangle functions $S(\alpha, \beta, \gamma; z)$ corresponding to the three cases considered above are the ones which give holomorphic solutions of the Chazy equation (2) via (16) and (17). These results are summarized in Table 1 below. Recall that $a_1 + a_2 + a_3 := p = 6$, and that $y = 3\phi'(z)/\phi(z)$ as given in (19). The automorphic groups Γ corresponding to the triangle functions are listed in the second column of the table. With the exception of Case 5, all others are subgroups of the modular group $\Gamma(1)$ (see e.g., [36] for notation); these will be discussed in the following section. The triangle function in Case 5 is not a simple

automorphic function of a Fuchsian group of first kind since the exponent difference at the vertex $z(0)$ given by $\beta = \frac{2}{3}$ is *not* reciprocal of a positive integer. Hence $S(0, \frac{2}{3}, 0; z)$ is not a single valued function of z on the domain D. Nevertheless, one can show from the conformal mapping properties of the triangle functions that the single-valued function $S(0, \frac{1}{2}, \frac{1}{3}; z)$ can be expressed as a degree-2 rational function of $S(0, \frac{2}{3}, 0; z)$ [29], namely

$$S(0, \tfrac{1}{2}, \tfrac{1}{3}; \epsilon z) = \frac{-4S(0, \frac{2}{3}, 0; z)}{[S(0, \frac{2}{3}, 0; z) - 1]^2},$$

with $\epsilon = \sqrt[3]{-\frac{1}{4}}$. Then $\phi(z)$ in case 5 is a constant multiple of the $\phi(\epsilon z)$ in case 1; hence, $y(z)$ obtained from $S(0, \frac{2}{3}, 0; z)$ is a single-valued function.

Case	Γ	$s(z) = S(\alpha, \beta, \gamma; z)$	$y = a_1w_1 + a_2w_2 + a_3w_3$	$\phi(z) = s'(z)/(s-1)^{b_1}s^{b_2}$
1	$\Gamma(1)$	$S(0, \frac{1}{2}, \frac{1}{3}; z)$	$y = 3w_1 + w_2 + 2w_3$	$\phi = s'/(s-1)^{1/2}s^{5/6}$
2	Γ^2	$S(0, \frac{1}{3}, \frac{1}{3}; z)$	$y = 2w_1 + 2w_2 + 2w_3$	$\phi = s'/(s-1)^{2/3}s^{2/3}$
3	$\Gamma_0(2)$	$S(0, \frac{1}{2}, 0; z)$	(i) $y = 3w_1 + 2w_2 + w_3$ (ii) $y = 3w_1 + w_2 + 2w_3$	(i) $\phi = s'/(s-1)^{1/2}s^{2/3}$ (ii) $\phi = s'/(s-1)^{1/2}s^{5/6}$
4	$\Gamma_0(3)$	$S(0, \frac{1}{3}, 0; z)$	(i) $y = 2w_1 + 3w_2 + w_3$ (ii) $y = 2w_1 + w_2 + 3w_3$	(i) $\phi = s'/(s-1)^{2/3}s^{1/2}$ (ii) $\phi = s'/(s-1)^{2/3}s^{5/6}$
5		$S(0, \frac{2}{3}, 0; z)$	$y = 4w_1 + w_2 + w_3$	$\phi = s'/(s-1)^{1/3}s^{5/6}$
6a	$\Gamma_0(4)$	$S(0, 0, 0; z)$	$y = w_1 + w_2 + 4w_3$	$\phi = s'/(s-1)^{5/6}s^{5/6}$
6b	$\Gamma(2)$	$S(0, 0, 0; z/2)$	$y = 2(w_1 + w_2 + w_3)$	$\phi = s'/(s-1)^{2/3}s^{2/3}$

Table 1: Triangle functions associated with the Chazy solution $y(z)$

Note that in each of the cases 3 and 4, the two different forms of $y(z)$ can be transformed to each other by interchanging w_2 and w_3. This transformation stems from the permutation of the two parabolic vertices $z(0)$ and $z(\infty)$ of the fundamental triangle T, thereby inducing the inversion map $s \to s^{-1}$ of the function field $s(z)$. Under this involution, $\phi(z)$ corresponding to the sub cases (i) and (ii) are constant multiple of each other, for both cases 3 and 4. In case 6, there are two automorphism groups which are conjugate to each other: $\Gamma_0(4) = g\Gamma(2)g^{-1}$ where $g = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$, and the fundamental domain of $\Gamma(2)$ is mapped under $z \to 2z$ to the fundamental domain of $\Gamma_0(4)$. The canonical automorphic function for $\Gamma(2)$ is the elliptic modular function $\lambda(z)$, which is related to the triangle function of $\Gamma_0(4)$ as $S(0, 0, 0; z) = \lambda(2z)[\lambda(2z) - 1]^{-1}$. From the above relation, it turns out that the $\phi(z)$ in case 6b can be transformed to that in case 6a due to the well-known (see e.g., [29]) functional relation: $\lambda(z) = 4\sqrt{\lambda(2z)}[1 + \sqrt{\lambda(2z)}]^{-2}$. Thus we see that there are many different representations of the solution of the Chazy equation in terms of Schwarzian triangle functions which in turn satisfy Schwarzian equations (13). As we have seen, each of these equations can be linearized via (9). We turn to this topic next.

4. Parametrization of the Chazy solutions

In the previous section, explicit solutions of the Chazy equation were presented in terms of the triangle functions listed in Table 1. The standard expressions of the triangle functions are usually via the theta or Dedekind's eta functions which admit Fourier or q-series expansions (as in (12a) or (12b)) developed in the neighborhood of the parabolic vertex

$z_o = i\infty$ (see e.g., [26] and references therein). It is however more convenient to express the Chazy solution $y(s(z))$ *implicitly*, that is, in terms of the variable s and a solution $u(s)$ of the linear equation (9). It is then possible to treat the nonlinear Chazy equation purely on the basis of the classical theory of linear Fuchsian differential equations. This is the main purpose of the present section. Yet another motivation is to relate our results to Ramanujan's work on the various parametrization of his functions P, Q, R, as mentioned in the introduction.

In this section, the domain of the automorphic functions $s(z)$ will be taken as $\mathrm{D} = \mathcal{H}$, the upper-half complex plane, and the hypergeometric form of the Fuchsian differential equation (9) will be considered, in order to make contact with standard literature. If $u(s)$ is a solution of (9), then the function

$$\chi(s) = s^{(\alpha-1)/2}(s-1)^{(\beta-1)/2}u(s) \tag{25}$$

satisfies the hypergeometric equation

$$\chi'' + \Big(\frac{1-\alpha}{s} + \frac{1-\beta}{s-1}\Big)\chi' + \frac{(\alpha+\beta-1)^2-\gamma^2}{4s(s-1)}\chi = 0\,, \tag{26a}$$

which can be written in more standard form as

$$s(s-1)\chi'' + [(a+b+1)s-c]\chi' + ab\chi = 0, \tag{26b}$$

where $a = (1-\alpha-\beta-\gamma)/2$, $b = (1-\alpha-\beta+\gamma)/2$, and $c = 1-\alpha$. The transformation (25) sets one of the exponents to 0 at each of the singular points $s = 0, 1$, but the exponent differences as well as the ratio $z(s)$ of any two linearly independent solutions, remain the same as those in (9). Consequently, the conformal mapping and the construction of the triangle function described in Section 2 can be carried out in an identical manner by employing the classical theory of the hypergeometric equation instead of (9). Let $\{\chi_1, \chi_2\}$ be a pair of linearly independent solutions of (26a) or (26b), and set $z(s) = \chi_2(s)/\chi_1(s)$, then $s'(z) = 1/z'(s) = \chi_1^2/W(\chi_1,\chi_2)$ where $W(\chi_1,\chi_2) = Cs^{\alpha-1}(s-1)^{\beta-1}$ is the Wronskian, and $C \neq 0$ is a constant depending on the chosen pair of solutions $\{\chi_1, \chi_2\}$. Use of this expression for $s'(z)$ in (19) provides an implicit parametrization for $y(z)$ in terms of χ_1 and its s-derivative, given by

$$y(z(s)) = \frac{3}{C}s^{-\alpha}(s-1)^{-\beta}\Big(2s(s-1)\chi_1\chi_1' - [(\tilde{b}_1+\tilde{b}_2)s - \tilde{b}_2]\chi_1^2\Big)\,, \tag{27}$$

with $\tilde{b}_1 = b_1 + \beta - 1$ and $\tilde{b}_2 = b_2 + \alpha - 1$. In the following, we construct the triangle functions $s(z)$ for all the cases listed in Table 1 as well as the corresponding Chazy solution $y(z)$ using (27). In each case, the pair of hypergeometric solutions $\{\chi_1, \chi_2\}$ are so chosen that the conformal map results in a fundamental region T which has the parabolic vertex $z(0) = i\infty \in \mathcal{H}$, around which suitable q-expansions for $s(z)$ are developed. In addition, one imposes boundary conditions on y, y', y'' in a consistent manner at this vertex in order to uniquely fix a special solution $y(z)$, which is then shown to be related to Ramanujan's $P(q)$ via (5). The ensuing results then relate to Ramanujan's parametrization for his theories of signatures 2,3,4, and 6, as mentioned in Section 1. To the best of our knowledge the cases from Table 1 that were not recorded by Ramanujan, correspond to Case 1 which was known by Chazy himself [2, 3], and Case 2. The representation of $y(z)$ via the triangle function $S(0, \frac{1}{3}, \frac{1}{3}; z)$ in case 2 can be found in Ref. [34].

4.1. Case 1

Here $\alpha = 0, \beta = \frac{1}{2}$ and $\gamma = \frac{1}{3}$. The corresponding group of automorphisms, Γ, is generated by rotations about the vertices $z(1)$ and $z(\infty)$ by π and $2\pi/3$, respectively, and a parabolic transformation stabilizing the vertex $z(0)$. It is known (see e.g. [28]) that the projective action of Γ is isomorphic to that of the modular group $\Gamma(1) := \mathrm{SL}_2(\mathbb{Z})$ acting on the upper-half plane $\mathcal{H}$ via fractional linear transformations according to

$$z \to \gamma(z) = \frac{az+b}{cz+d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

and which is generated by the fundamental transformations:

$$z \to z+1 \quad (\text{translation}), \qquad z \to \frac{-1}{z} \quad (\text{inversion}).$$

It is also customary to choose the fundamental triangle T in $\mathcal{H}$ such that it is a strip parallel to the imaginary axis, bounded by the unit circle $|z| = 1$, and the vertices are located at $z(0) = i\infty$, $z(1) = i$, $z(\infty) = e^{2\pi i/3}$. To construct the triangle function $S(0, \frac{1}{2}, \frac{1}{3}; z)$ on T, one starts with the hypergeometric equation (cf. (26a), (26b))

$$\chi'' + \left(\frac{1}{s} + \frac{1}{2(s-1)}\right)\chi' + \frac{5/144}{s(s-1)}\chi = 0, \tag{28}$$

with parameters $a = \frac{1}{12}$, $b = \frac{5}{12}$ and $c = 1$. Equation (28) admits a one-dimensional space of single-valued solutions spanned by the hypergeometric function ${}_2F_1(\frac{1}{12}, \frac{5}{12}; 1; s)$ that is holomorphic in a neighborhood of $s = 0$ and is normalized to unity there. The second independent solution of (28) is chosen as ${}_2F_1(\frac{1}{12}, \frac{5}{12}; \frac{1}{2}; 1-s)$, which is holomorphic in a neighborhood of $s = 1$. Next, define the conformal map $z(s)$ as follows

$$z(s) = \frac{\chi_2}{\chi_1}, \qquad \chi_1 = {}_2F_1(\tfrac{1}{12}, \tfrac{5}{12}; 1; s), \quad \chi_2 = A\, {}_2F_1(\tfrac{1}{12}, \tfrac{5}{12}; \tfrac{1}{2}; 1-s) + B\chi_1, \tag{29}$$

where A, B are constants to be determined by fixing the vertices of T as specified above. The analytic continuation of ${}_2F_1(\frac{1}{12}, \frac{5}{12}; \frac{1}{2}; 1-s)$ into the neighborhood of $s = 0$ is given by (see e.g. [35]),

$${}_2F_1(\tfrac{1}{12}, \tfrac{5}{12}; \tfrac{1}{2}; 1-s) = \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{12})\Gamma(\frac{5}{12})}\left[-\log(s)\,{}_2F_1(\tfrac{1}{12}, \tfrac{5}{12}; 1; s) + \sum_{n=0}^{\infty} \frac{(\frac{1}{12})_n(\frac{5}{12})_n}{(n!)^2} h_n s^n\right],$$

where $\Gamma(\cdot)$ are Gamma functions, $h_n := 2\psi(1+n) - \psi(\frac{1}{12}+n) - \psi(\frac{5}{12}+n)$, and $\psi(\cdot) := \Gamma'(\cdot)/\Gamma(\cdot)$ are the Digamma functions. Hence from (29),

$$z(s) = A\,\frac{{}_2F_1(\frac{1}{12}, \frac{5}{12}; \frac{1}{2}; 1-s)}{{}_2F_1(\frac{1}{12}, \frac{5}{12}; 1; s)} + B = A_1(-\log(s) + h(s)) + B, \qquad A_1 = A\,\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{12})\Gamma(\frac{5}{12})},$$

near $s = 0$, and $h(s)$ is holomorphic in that neighborhood. The constant A is determined by fixing $A_1 = i/2\pi$ above, so that $z(s) \to i\infty$ as $s \to 0^+$, and $z \to z+1$ onto the next branch as s makes a circuit around $s = 0$. The constant B can then be determined by demanding that $\lim_{s\to 1} z(s) = i$. This sets the elliptic vertex $z(1) = i$, and yields $B = -i$

from the expression of $z(s)$ above, after using ${}_2F_1(\frac{1}{12},\frac{5}{12};1;1) = \Gamma(\frac{1}{2})[\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})]^{-1}$ and the formula $\Gamma(x)\Gamma(1-x) = \pi\csc(\pi x)$. Consequently, the function $q := e^{2\pi iz}$ has the power series representation $q = se^{-h(s)+2\pi} = B_1 s(1 + a_1 s + a_2 s^2 + \ldots)$, which can be inverted to obtain a q-series representation of the triangle function $s(z) = S(0,\frac{1}{2},\frac{1}{3};z)$ in the neighborhood of the parabolic vertex $q = 0\,(z = i\infty)$. In particular, the constant $B_1 = e^{2\pi - h(0)}$ can be evaluated from the the analytic continuation formula of ${}_2F_1(\frac{1}{12},\frac{5}{12};\frac{1}{2};1-s)$ given above. One easily checks that $h(0) = h_0 := 2\psi(1) - \psi(\frac{1}{12}) - \psi(\frac{5}{12})$ which evaluates to $h_0 = 2\pi + 3\log 12$ using Gauss's Digamma formula (see [35]), and thus $B_1 = 1/1728$. The first few terms of the q-expansion of the triangle function is then given by

$$S(0,\tfrac{1}{2},\tfrac{1}{3};z) = 1728q(1 - 744q + 356652q^2 + \ldots)\,.$$

Using the analytic continuations of the pair $\{\chi_1,\chi_2\}$ to a neighborhood of $s = \infty$, and from the values of the constants A, B obtained above, it can be verified that $z(\infty) = e^{2\pi i/3}$. We note here that $S(0,\frac{1}{2},\frac{1}{3};z)$ is the reciprocal of the well known J invariant associated with the modular group $\Gamma(1)$.

In order to derive the implicit parametrization $y(z(s))$ for the Chazy solution, one needs to calculate the Wronskian $W(\chi_1,\,\chi_2)$ of the two solutions specified in (29), of the hypergeometric equation (28). A short calculation employing the analytic continuation formula for ${}_2F_1(\frac{1}{12},\frac{5}{12};\frac{1}{2};1-s)$ near $s = 0$ gives

$$W(\chi_1,\,\chi_2) = \frac{i}{2\pi}[W(\chi_1,\,h(s)) - \frac{\chi_1^2}{s}] = Cs^{-1}(s-1)^{-\frac{1}{2}}\,,$$

where $h(s)$ is analytic near $s = 0$, and the last expression on the right follows from Abel's formula. Hence, letting $s \to 0^+$ in above, yields the constant $C = 1/2\pi$. Then from (27) with $b_1 = \frac{1}{2}$ and $b_2 = \frac{5}{6}$ (cf. Table 1), the following parametrization is obtained

$$y(z(s)) = \pi i(1-s)^{\frac{1}{2}}(\chi_1^2 + 12s\chi_1\chi_1')\,, \qquad \chi_1(s) = {}_2F_1(\tfrac{1}{12},\tfrac{5}{12};1;s)\,. \tag{30}$$

Recall from Section 1 that Ramanujan's modular function $P(q)$ introduced in (3) is related to the Chazy solution via $y(z) = \pi iP(q)$. Therefore, from (30) it is possible to obtain an implicit parametrization for Ramanujan's P,Q,R in terms of the solutions of the hypergeometric equation (28).

Proposition 4. *Let $z(s)$ be the quotient of hypergeometric solutions χ_1 and χ_2 defined in* (29)*, and $q = e^{2\pi iz(s)}$, then*

$$P(q) = (1-s)^{\frac{1}{2}}(\chi_1^2 + 12s\chi_1\chi_1')\,, \qquad Q(q) = \chi_1^4\,, \qquad R(q) = (1-s)^{\frac{1}{2}}\chi_1^6\,.$$

Proof. First, from (3) note that the Ramanujan functions satisfy the conditions $P \to 1$, $Q \to 1$, $R \to 1$ as $q \to 0$. Next, from (2) and using the forms f_2, f_3 from Lemma 1 it is easy to verify that the triple $\{y/\pi,\ \frac{6}{\pi^2}f_2,\ \frac{9}{(i\pi)^3}f_3\}$ satisfies Ramanujan's differential system (4). On the other hand, it follows from (30) that $y/\pi i \to 1$ as $s \to 0$ (equivalently, $q \to 0$). Moreover, differentiating $y(z)$ in (30) successively and using $s'(z) = \chi_1^2/W(\chi_1,\chi_2)$, one obtains the expressions $\frac{6}{\pi^2}f_2 = \chi_1^4$ and $\frac{9}{(i\pi)^3}f_3 = \sqrt{1-s}\,\chi_1^6$, which satisfy the *same* conditions as Q and R, respectively when $q \to 0$. Therefore, uniqueness of solutions of the differential system (4) yields the desired result. $\square$

Substituting the q-expansion for $S(0, \frac{1}{2}, \frac{1}{3}; z)$ into the hypergeometric series for χ_1, it is possible to recover from the above parametrizations the q-series for P, Q, R in (3). On the other hand, Proposition 4 provides an elegant representation for the the triangle function $S(0, \frac{1}{2}, \frac{1}{3}; z)$ as well as a remarkable identity, namely

$$S(0, \tfrac{1}{2}, \tfrac{1}{3}; z) = \frac{Q^3 - R^2}{Q^3}, \qquad Q^{\frac{1}{4}} = {}_2F_1(\tfrac{1}{12}, \tfrac{5}{12}; 1; \tfrac{Q^3-R^2}{Q^3}).$$

The first expression can be used together with (3) to obtain the q-expansion for $S(0, \frac{1}{2}, \frac{1}{3}; z)$ derived above.

The transformation property (18) for $y(z)$ under the action of the modular group $\Gamma = \Gamma(1)$ implies that $P = y/\pi i$ is a *quasi-modular* form of weight 2 and affinity coefficient $p = 6$ on $\Gamma(1)$. Moreover, by comparing (4) with the result of Lemma 1, it follows that $Q = \frac{6}{\pi^2} f_2$ and $R = \frac{9}{(\pi i)^3} f_3$ are *modular* forms of weight 4 and 6 respectively, on $\Gamma(1)$. In fact, $P = E_2$, $Q = E_4$ and $R = E_6$, where $E_k(q)$ is the normalized Eisenstein series of weight k for the modular group $\Gamma(1)$. E_k is a holomorphic modular form, and is defined (for even positive integer k) as

$$E_k(q) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \qquad q = e^{2\pi i z}, \ z \in \mathcal{H}, \tag{31}$$

where B_k is the kth Bernoulli number and $\sigma_k(n)$ is the sum-of-divisor function introduced in Section 1. Detailed discussions of modular forms appear in several monographs (see e.g. [36, 37]. The vector space of holomorphic modular forms of weight k on $\Gamma(1)$ is denoted by $M_k(\Gamma(1))$. Proposition 4 provides a parametrization for any $f \in M_k(\Gamma(1))$ since it is known from the theory of modular forms that f belongs to the polynomial ring $\mathbb{C}[E_4, E_6]$. For $4 \leq k \leq 10$, $M_k(\Gamma(1))$ is one-dimensional, and is spanned by the Eisenstein series E_k. It turns out that the left hand side of (2) can be expressed as $f_4 + 4f_2^2$ (see Section 3) which by virtue of Lemma 1, must be in $M_8(\Gamma(1))$. Hence, $f_4 + 4f_2^2 = \mathbb{C}E_8$. But with $y = \pi i E_2$, the expression $f_4 + 4f_2^2$ is a differential polynomial in E_2, and vanishes as $q \to 0$, whereas $E_8 \to 1$. Therefore, the Chazy equation $f_4 + 4f_2^2 = 0$ follows from this "modular" argument.

4.2. Case 2

The triangle group associated with $\alpha = 0$, $\beta = \frac{1}{3}$, $\gamma = \frac{1}{3}$ is denoted by Γ^2. It is a normal subgroup of the modular group $\Gamma(1)$ of index 2, generated by the period-3 transformations

$$T_1: \quad z \to \frac{-1}{z+1}, \qquad T_2: \quad z \to \frac{z-1}{z},$$

whose respective fixed points $e^{2\pi i/3}$ and $e^{\pi i/3}$ in $\mathcal{H}$ form the vertices $z(1)$ and $z(\infty)$ of the fundamental triangle T of Γ^2 together with the parabolic fixed point $z(0) = i\infty$. An important point to note here is that the stabilizer of the vertex $z(0) = i\infty$ in Γ^2 is given by the transformation $T_2T_1: \ z \to z+2$ instead of the translation $z \to z+1$, which is *not* an element of Γ^2.

One proceeds with the construction of the triangle function $S(0, \frac{1}{3}, \frac{1}{3}; z)$ in a similar manner as in Case 1. Define the conformal mapping by

$$z(s) = \frac{\chi_2}{\chi_1}, \qquad \chi_1 = {}_2F_1(\tfrac{1}{6}, \tfrac{1}{2}; 1; s), \quad \chi_2 = A \ {}_2F_1(\tfrac{1}{6}, \tfrac{1}{2}; \tfrac{2}{3}; 1-s) + B\chi_1, \tag{32}$$

where ${}_2F_1(\frac{1}{6},\frac{1}{2};1;s)$, ${}_2F_1(\frac{1}{6},\frac{1}{2};\frac{2}{3};1-s)$ are solutions holomorphic in the neighborhoods of $s=0$ and $s=1$ respectively, of the hypergeometric equation

$$\chi'' + \left(\frac{1}{s} + \frac{2}{3(s-1)}\right)\chi' + \frac{1/12}{s(s-1)}\chi = 0\,, \tag{33}$$

with parameters $a=\frac{1}{6}$, $b=\frac{1}{2}$ and $c=1$. The constants A, B are determined as before by considering the analytic continuation of the solutions near $s=0$, where

$${}_2F_1(\tfrac{1}{6},\tfrac{1}{2};\tfrac{2}{3};1-s) = \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{6})\Gamma(\frac{1}{2})}\left[-\log(s){}_2F_1(\tfrac{1}{6},\tfrac{1}{2};1;s) + \sum_{n=0}^{\infty}\frac{(\frac{1}{6})_n(\frac{1}{2})_n}{(n!)^2}\,h_n s^n\right],$$

with $h_n := 2\psi(1+n) - \psi(\frac{1}{6}+n) - \psi(\frac{1}{2}+n)$. Then from (32), one has

$$z(s) = A_1(-\log(s) + h(s)) + B\,, \qquad A_1 = A\,\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{6})\Gamma(\frac{1}{2})}\,,$$

where $h(s)$ is a holomorphic function near $s=0$. As indicated earlier, since $z \to z+2$ stabilizes the vertex at $z(0)=i\infty$, the pair $\{\chi_1,\,\chi_2\}$ must form a basis of solutions near $s=0$ with monodromy $\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)$. Thus, the constant A is determined by taking $A_1 = i/\pi$ above, so that $z \to z+2$ onto the next branch as s makes a circuit around $s=0$. The constant B is then obtained from the condition $z(1)=e^{2\pi i/3}$, which yields $B=-e^{\pi i/3}$ after a similar calculation as in case 1. With these values of A, B in (32), it is possible to construct the power series for $q := e^{\pi i z(s)}$ near $s=0$, and its inverse $s(z) = B_2 q(1 + b_1 q + b_2 q^2 + \ldots)$ which gives the q-series for the triangle function $S(0,\frac{1}{2},\frac{1}{3};z)$. The constant $B_2 = i48\sqrt{3}$, which follows from the leading coefficient $h_0 = 2\psi(1) - \psi(\frac{1}{6}) - \psi(\frac{1}{2})$ above and evaluation of Digamma functions at rational arguments.

From the Wronskian $W(\chi_1,\,\chi_2)$ of the two solutions of (33), one computes the constant $C = A_1 = i/\pi$ in (27). With this value of C, and $b_1 = b_2 = \frac{2}{3}$ from Table 1, (27) then gives the parametrization

$$y(z(s)) = \pi i P = \pi i(1-s)^{\frac{2}{3}}(\chi_1^2 + 6s\chi_1\chi_1')\,, \qquad \chi_1(s) = {}_2F_1(\tfrac{1}{6},\tfrac{1}{2};1;s)\,. \tag{34}$$

One also deduces from arguments aligned with Proposition 4 that,

$$Q(q) = (1-s)^{\frac{1}{3}}\chi_1^4\,, \qquad R(q) = (1-\tfrac{1}{2}s)\chi_1^6\,, \qquad q = e^{2\pi i z}\,.$$

4.3. Cases 3, 4 & 6

The automorphism groups in these cases correspond to the level-N congruence subgroups $\Gamma_0(N)$ of the modular group $\Gamma(1)$, for $N = 2,\,3,\,4$. The congruence subgroup $\Gamma_0(N)$ is defined by

$$\Gamma_0(N) := \left\{\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod N\right\}.$$

The fundamental triangle T in each of these 3 cases has 2 parabolic vertices at $z(0) = i\infty$, $z(\infty) = 0$, while the vertex $z(1)$ corresponds to an elliptic fixed point of order 2 at $z(1) = i$ for $\Gamma_0(2)$, an elliptic fixed point of order 3 at $z(1) = e^{2\pi i/3}$ for $\Gamma_0(3)$, and a

parabolic point at $z(1) = \frac{1}{2}$ for $\Gamma_0(4)$. Recall from Section 3 that in Case 6(b), the automorphism group is $\Gamma(2)$ which is conjugate to $\Gamma_0(4)$. The principal congruence subgroup $\Gamma(2)$ is defined as

$$\Gamma(2) := \left\{ \gamma \in P\mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2 \right\} ,$$

which is a normal subgroup of $\Gamma(1)$ of index 6. The fundamental triangle of $\Gamma(2)$ is same as that of $\Gamma_0(4)$, but with respect to the variable $z/2$. That is, the parabolic vertices are located at $z(0) = i\infty, z(\infty) = 0$, and $z(1) = 1$ with respect to the local coordinate $z \in \mathcal{H}$.

The triangle function $S(0, \beta, 0; z)$ in each of these cases will be constructed by developing a q-expansion near the parabolic vertex $z(0) = i\infty$, which is stabilized by the translation $z \to z + 1$. From (26a), the hypergeometric equation corresponding to exponent differences $0, \beta, 0$ is

$$\chi'' + \Big(\frac{1}{s} + \frac{1-\beta}{s-1}\Big)\chi' + \frac{(\beta-1)^2}{4s(s-1)}\chi = 0\,, \tag{35}$$

with parameters $a = b = (1-\beta)/2$, and c=1. So the exponents are 0, 0 at the regular singular point $s = 0$, and a, a at $s = \infty$. A one-dimensional space of solutions for (35) near $s = 0$ and $s = \infty$ respectively, are spanned by the functions ${}_2F_1(a, a; 1; s)$ and $(-s)^{-a}\,{}_2F_1(a, a; 1; s^{-1})$. The second linearly independent solution near each singular point contains logarithms. The conformal mapping is defined as

$$z(s) = A\frac{\chi_2}{\chi_1}\,, \qquad \chi_1 = {}_2F_1(a, a; 1; s)\,, \quad \chi_2 = (-s)^{-a}\ {}_2F_1(a, a; 1; s^{-1})\,, \tag{36}$$

where the constant A is determined by considering the analytic continuation of χ_2 near $s = 0$. This follows from the formula (see e.g. [35]),

$${}_2F_1(a, a; 1; s^{-1}) = \frac{\sin(\pi a)}{\pi}(-s)^a \left[-\log(-s){}_2F_1(a, a; 1; s) + \sum_{n=0}^{\infty} \frac{(a)_n^2}{(n!)^2}\, h_n s^n \right] ,$$

$-\pi < \arg(-s) < \pi$, and $h_n := 2\psi(1+n) - \psi(a+n) - \psi(1-a-n)$. Proceeding analogously as in the previous two cases, $z(s)$ in (32) takes the form

$$z(s) = A_1(-\log(-s) + h(s))\,, \qquad A_1 = A\,\frac{\sin(\pi a)}{\pi}\,,$$

and $h(s)$ is holomorphic near $s = 0$. The constant A (see Table 2) is determined by taking $A_1 = i/2\pi$ above for cases 3, 4, and 6a, such that $\{\chi_1, \chi_2\}$ has monodromy $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ at $s = 0$. Finally, inverting the power series for $q := e^{2\pi i z(s)}$ leads to the expansion $s(z) = \widehat{B}q(1 + b_1 q + b_2 q^2 + \ldots)$ for $S(0, \beta, 0; z)$. The value of the constant $\widehat{B} = -e^{-h_0}$ with $h_0 = 2\psi(1) - \psi(a) - \psi(1-a)$, is listed in Table 2 below for each group $\Gamma_0(N)$. For case 6b corresponding to the group $\Gamma(2)$, the parabolic vertex $z(0) = i\infty$ is stabilized by the translation $z \to z + 2$. Thus one needs to choose $A_1 = i/\pi$ above, which leads to $A = i$ and $q = e^{\pi i z(s)}$ in this case.

We remark that the triangle functions $S(0, \beta, 0; z)$ associated with $\Gamma_0(N)$, $N = 2, 3, 4$ as well as $\Gamma(2)$ can be expressed compactly in terms of Dedekind eta functions. In fact,

N	β	$a=(1-\beta)/2$	$A=i\csc(\pi a)/2$	$\widehat{B}$
2	$\frac{1}{2}$	$\frac{1}{4}$	$i/\sqrt{2}$	-64
3	$\frac{1}{3}$	$\frac{1}{3}$	$i/\sqrt{3}$	-27
4	0	$\frac{1}{2}$	$i/2$	-16

Table 2: parameters for constructing the triangle functions $S(0,\beta,0;z)$.

such expressions are available for the automorphic functions associated with all subgroups $\Gamma_0(N)$ which are of genus zero, see Ref [26] for a complete list.

For the cases 3, 4 and 6a, the Wronskian of the two solutions in (36) is given by $W(\chi_1,\,\chi_2)=Cs^{-1}(s-1)^{-2a}$, where $C=(-1)^{2a}/2\pi i$. Then from (27) one obtains the parametrization,

$$y(z(s))=-6\pi i(1-s)^{2a-1}[\{(1+2a-b_1-b_2)s+(b_2-1)\}\chi_1^2+2s(s-1)\chi_1\chi_1'], \quad (37)$$

where $\chi_1(s)={}_2F_1(a,a;1;s)$ and (b_1,b_2) is given in Table 1. Note that in case 6b, the constant $C=i/\pi$ in the Wronskian function. The parametrization of the Ramanujan functions P,Q,R for all three cases are listed in Table 3. These are obtained in the same manner as outlined in Proposition 4. Note that there are two distinct parametrizations for each of the three cases corresponding to $\Gamma_0(N)$, $N=2,3,4$. For $\Gamma_0(2)$ or $\Gamma_0(3)$, the pair of parametrizations are related via the involution $s\to s^{-1}$, as noted below Table 1. Under this involution, it follows from (36) that $z\to\frac{A^2}{z}$ as the two linearly independent solutions χ_1 and χ_2 of (35) are switched in the quotient z. This is related (under appropriate normalization) to the Fricke involution: $z\to-1/Nz$ for $\Gamma_0(N)$ (see [26] for details). Thus in Table 3, the two parametrization in the case of $\Gamma_0(2)$ or $\Gamma_0(3)$ are transformed from one to the other by switching $\chi={}_2F_1(a,a;1;s)$ to $\chi=(-s)^{-a}\,{}_2F_1(a,a;1;s)$, $s\to s^{-1}$ as well as, taking into account the sign reversal in the Wronskian W appearing in (27) through the chain rule formula $d/dz=s'(z)d/ds=(\chi^2/W)d/ds$.

Γ	(b_1,b_2)	$P=y/\pi i$	Q	R
$\Gamma_0(2)$	$(\frac{1}{2},\frac{2}{3})$	$2(1-s)^{1/2}(\chi^2+6s\chi\chi')$	$(4-s)\chi^4$	$(1-s)^{1/2}(s+8)\chi^6$
$\Gamma_0(2)$	$(\frac{1}{2},\frac{5}{6})$	$(1-s)^{1/2}(\chi^2+12s\chi\chi')$	$(1-4s)\chi^4$	$(1-s)^{1/2}(8s+1)\chi^6$
$\Gamma_0(3)$	$(\frac{2}{3},\frac{1}{2})$	$3(s-1)^{2/3}(\chi^2+4s\chi\chi')$	$(s-1)^{1/3}(s-9)\chi^4$	$(27-18s-s^2)\chi^6$
$\Gamma_0(3)$	$(\frac{2}{3},\frac{5}{6})$	$(s-1)^{2/3}(\chi^2+12s\chi\chi')$	$(s-1)^{1/3}(9s-1)\chi^4$	$(1+18s-27s^2)\chi^6$
$\Gamma_0(4)$	$(\frac{5}{6},\frac{5}{6})$	$(1-2s)\chi^2-12s(s-1)\chi\chi'$	$16(s^2-s+\frac{1}{16})\chi^4$	$32(2s-1)(s^2-s-\frac{1}{32})\chi^6$
$\Gamma(2)$	$(\frac{2}{3},\frac{2}{3})$	$(1-2s)\chi^2-6s(s-1)\chi\chi'$	$(s^2-s+1))\chi^4$	$\frac{1}{2}(2s-1)(s^2-s-2)\chi^6$

Table 3: Parametrization of P,Q,R. Here $\chi={}_2F_1(a,a;1;s)$ and $s(z)=S(0,\beta,0;z)$ with a and β values given in Table 2.

The relationship between the parametrization of the Ramanujan functions (Eisenstein series) P,Q,R listed in Table 3 and Ramanujan's theories of elliptic integrals will now be established. As mentioned in Section 1, Ramanujan originally gave the parametrization (7) in terms of complete elliptic integrals of the first kind. These correspond to the parametrization associated to the groups $\Gamma_0(4)$ and $\Gamma(2)$ in Table 3. The parametrization corresponding to $\Gamma_0(2)$ and $\Gamma_0(3)$ are related to the parametrization (8) in Ramanujan's

alternative theories. The hypergeometric functions appearing in (7) and (8) are related to those in Table 3 via the well-known Pfaff transformation

$$ {}_2F_1(a,b;c;s) = (1-s)^{-a}\, {}_2F_1(a,c-b;c;x(s))\,, \qquad x(s) = \frac{s}{s-1}\,, $$

so that in the present case with $a=b$, $c=1$, (36) takes the form

$$ z(s) = A(-s)^{-a}\frac{{}_2F_1(a,a;1;s^{-1})}{{}_2F_1(a,a;1;s)} = \frac{i}{2\sin(\pi a)}\frac{{}_2F_1(a,1-a;1;1-x)}{{}_2F_1(a,1-a;1;x)}\,. \tag{38} $$

Then by setting $a = 1/r$ and $2\pi i z(s) = -u_r$, (38) coincides with (8), and thus one recovers from Table 3 the Eisenstein series parametrizations in Ramanujan's theories for signatures $r = 4,3,2$ corresponding to the groups $\Gamma_0(2), \Gamma_0(3)$ and $\Gamma_0(4)$, respectively. For completeness, these are presented in Table 4 below, in terms of the hypergeometric function $\chi_r(x) := {}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; x)$ which appears in Ramanujan's theories.

4.4. Case 5

The triangle group corresponding to $\alpha = 0$, $\beta = \frac{2}{3}$, $\gamma = 0$ is not a Fuchsian group of first kind ($1/\beta$ is not a positive integer) although the triangle functions $s(z) = S(0, \frac{1}{2}, \frac{1}{3}; z)$ and $\widehat{s}(z) = S(0, \frac{2}{3}, 0; z)$ are related via $s(\epsilon z) = -4\widehat{s}(z)/(1-\widehat{s}(z))^2$, $\epsilon = -\sqrt[3]{\frac{1}{4}}$, as mentioned in Section 3. The main significance of this case lies in the fact that it corresponds to Ramanujan's alternative theory of signature $r=6$ as outlined below.

The triangle function $S(0, \frac{2}{3}, 0; z)$ can be constructed employing the hypergeometric theory in exactly the same way as for the cases associated with the subgroups $\Gamma_0(N)$, $N = 2,3,4$ of the modular group, even though this case is not known to have any "modular interpretation" [26]. Specifically, one proceeds from the hypergeometric equation (35) with $\beta = \frac{2}{3}$, and defines $z(s)$ as in (36) with $a = \frac{1}{6}$. Then one finds that the constants $A = i$ and $\widehat{B} = -e^{h_0} = -432$. The Wronskian in this case turns out to be $W(\chi_1, \chi_2) = Cs^{-1}(s-1)^{-1/3}$ where $C = -1/2\pi i$. Finally, substituting $(b_1, b_2) = (\frac{1}{3}, \frac{5}{6})$ from Table 1 into (37) and using arguments similar to that in Proposition 4, one obtains the following parametrization for the Eisenstein series in Ramanujan's theory of signature $r = a^{-1} = 6$,

$$ P(q) = \frac{y(z(s))}{\pi i} = (1-s)^{\frac{1}{3}}(\chi_1^2 + 12 s\chi\chi')\,, \qquad Q(q) = (1-s)^{\frac{2}{3}}\chi^4\,, \qquad R(q) = (1+s)\chi^6\,, $$
$$ \chi(s) = {}_2F_1(\tfrac{1}{6}, \tfrac{1}{6}; 1; s) \qquad q = \exp\Big(-2\pi(-s)^{-\frac{1}{6}}\,\frac{{}_2F_1(\frac{1}{6},\frac{1}{6};1;s^{-1})}{{}_2F_1(\frac{1}{6},\frac{1}{6};1;s)}\Big)\,. \tag{39} $$

The map in (38) induced by the Pfaff transformation also applies to this case, and leads to the alternative parametrization for $r=6$ listed in Table 4.

The parametrizations listed in Table 4 can be found elsewhere, e.g., in Refs. [24, 25]. The first entry in Table 4 for $r=2$ appears in (7) as Ramanujan's original parametrization. The second entry (case 2*) corresponds to the parametrization of the Eisenstein series in terms of the elliptic modular function $x = \lambda(z)$. Recall from Table 1 that the triangle function for $\Gamma(2)$ is $S(0,0,0;z/2)$ where $S(0,0,0;z)$ is the triangle function for $\Gamma_0(4)$. Then from the formula given at the end of Section 3, it follows that $S(0,0,0;z/2) = \lambda(z)[\lambda(z)-1]^{-1}$. The expression of $x(s)$ in the Pfaff transformation formula implies that the involution $s \to \frac{1}{s}$ corresponds to the involution $x \to 1-x$, as well as the switching of

r	(b_1, b_2)	$P = y/\pi i$	Q	R
2	$(\frac{5}{6}, \frac{5}{6})$	$(1-5x)\chi_2^2 + 12x(1-x)\chi_2\chi_2'$	$(1+14x+x^2)\chi_2^4$	$(1+x)(1-34x+x^2)\chi_2^6$
$2*$	$(\frac{2}{3}, \frac{2}{3})$	$(1-2x)\chi_2^2 + 6x(1-x)\chi_2\chi_2'$	$(1-x+x^2)\chi_2^4$	$(1+x)(1-\frac{5}{2}x+x^2)\chi_2^6$
3	$(\frac{2}{3}, \frac{1}{2})$	$(3-4x)\chi_3^2 + 12x(1-x)\chi_3\chi_3'$	$(9-8x)\chi_3^4$	$(8x^2-36x+27)\chi_3^6$
3	$(\frac{2}{3}, \frac{5}{6})$	$(1-4x)\chi_3^2 + 12x(1-x)\chi_3\chi_3'$	$(1+8x)\chi_3^4$	$(1-20x-8x^2)\chi_3^6$
4	$(\frac{1}{2}, \frac{2}{3})$	$(2-3x)\chi_4^2 + 12x(1-x)\chi_4\chi_4'$	$(4-3x)\chi_4^4$	$(8-9x)\chi_4^6$
4	$(\frac{1}{2}, \frac{5}{6})$	$(1-3x)\chi_4^2 + 12x(1-x)\chi_4\chi_4'$	$(1+3x)\chi_4^4$	$(1-9x)\chi_4^6$
6	$(\frac{1}{3}, \frac{5}{6})$	$(1-2x)\chi_6^2 + 12x(1-x)\chi_6\chi_6'$	χ_6^4	$(1-2x)\chi_6^6$

Table 4: Parametrization of P, Q, R via $\chi_r = {}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; x)$, $x(s) = s/(s-1)$. The corresponding triangle functions are $s(z) = S(0, \beta, 0; z)$ with $\beta = 1 - \frac{2}{r}$.

the two ${}_2F_1$ functions in the quotient $z(s)$ in (38). This leads to the similar transformation as in Table 3, between the two cases for $r = 3$ or 4 listed in Table 4. It is interesting to note that the parametrization in the sextic case remain invariant under the involution $x \to 1 - x$.

5. Concluding remarks

In this note, we have reviewed the relationship between the Chazy equation and the conformal maps defined by ratios of solutions of Fuchsian equations with 3 regular singular points. The Chazy equation turns out to be a particular case of more general third order nonlinear differential equations which can be derived systematically by developing the transformation properties of functions under the projective monodromy group associated with the Fuchsian equation. Much of this note has been devoted to exemplify the important connection between the Chazy equation and Ramanujan's study of elliptic integrals and theta functions which play a defining role in the contemporary theory of modular forms and elliptic surfaces. By the way of elucidating this beautiful relationship, we have derived all possible parametrizations of the Chazy solution $y(z)$ via the Schwarz triangle functions. We show that these parametrizations are also related to Ramanujan's parametrization for the Eisenstein series P, Q, R in terms of hypergeometric functions, arising in his theories of modular equations of signatures 2, 3, 4 and 6. We also give two additional hypergeometric parametrizations stemming from the triangle functions associated with the modular group $\Gamma(1)$ and its index 2 subgroup Γ^2. The case corresponding to the full modular group $\Gamma(1)$ was considered by Chazy himself [1]. Furthermore, we note that it is possible to systematically construct a number of third order nonlinear equations associated with modular as well as other automorphic groups. Some of these equations corresponding to the subgroups groups $\Gamma_0(2), \Gamma_0(3)$ and $\Gamma_0(4)$ of $\Gamma(1)$ have been studied recently in Refs. [38, 39] and in Ref. [27] which analyzes the Ramanujan's differential system (4) as mentioned in Section 1. We end this note with an application of the parametrization of Eisenstein series in Table 4 to solutions of differential equations.

Consider the DH system of differential equations which was introduced in Section 2, and which corresponds to the system (15) with $\tau = 0$. As mentioned there, if the variables w_j, $j = 1, 2, 3$ satisfy the DH system then $y(z) := 2(w_1 + w_2 + w_3)$ solves the Chazy equation (2). In the process of verifying this assertion, one computes $y'(z) = 2(w_1w_2 + w_2w_3 + w_3w_1)$ and $y''(z) = 12w_1w_2w_3$, which implies that the DH variables w_j are the distinct roots of the cubic $w^3 - (y/2)w^2 + (y'/2)w - y''/12 = 0$. Expressing y, y', y''

in terms of the functions P, Q, R (see e.g. Proposition 4) and introducing $t = (6/i\pi)w$, one obtains the cubic equation

$$t^3 - 3Pt^2 + 3(P - Q^2)t - (P^3 - 3PQ + 2R) = 0\,,$$

whose coefficients are polynomials in P, Q, R. Now note from Table 1 that $y(z) := 2(w_1 + w_2 + w_3)$ corresponds to the case 6b associated with the group $\Gamma(2)$. The hypergeometric parametrization for this is the case r=2* in Table 4. Using those parametrizations for P, Q, R in the above cubic, one can verify that the cubic factorizes as $(t-t_1)(t-t_2)(t-t_3)$ with

$$t_1 = 3(1-x)\chi_2^2 + t_3\,, \qquad t_2 = -3x\chi_2^2 + t_3\,, \qquad t_3 = 6x(1-x)\chi_2\chi_2'\,.$$

Thus one immediately recovers the special solutions $w_j = (\pi i/6)t_j$, $j = 1, 2, 3$ for the DH system in terms of elliptic modular form $x = \lambda(z)$ and $\chi_2 = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x) = (2/\pi)K(x)$, where $K(x)$ is the complete integral of the first kind defined in Section 1. Such linearization procedures to derive exact solutions can be also employed to a large class of nonlinear differential equations with automorphic properties as discussed in this article. These problems will be addressed elsewhere.

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